# Math 246A Lecture 17 Notes

Daniel Raban

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## 1 Simply Connected Domains and Cauchy's theorem

### 1.1 Simply connected domains

**Definition 1.1.** A cycle $\gamma \subseteq \Omega$ is **homologous to 0** if $n(\gamma, z) = 0$ for all $z \notin \Omega$.

We write $\gamma \sim 0$. We also say that $\gamma \sim \gamma_2$ if $\gamma_1 - \gamma_2 \sim 0$, which is iff $n(\gamma_1, z) = n(\gamma_2, z)$ for al $z \notin \Omega$.

**Theorem 1.1** (Cauchy's theorem, general form)**.** *Let $\Omega$ be a domain and $\gamma \subseteq \Omega$ be a $C^1$ cycle. If $\gamma \sim 0$, then*

$$\int_\gamma f(z)\, dz = 0$$

*for all $g \in H(\Omega)$.*

We can also restate this with 1-forms.

**Definition 1.2.** A 1-form $P\, dx + Q\, dy$ is **closed** if $P, Q \in C^1$, $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$, $\frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$.

**Theorem 1.2.** *Let $\Omega$ be a domain and $\gamma \subseteq \Omega$ be a $C^1$ cycle. If $\gamma \sim 0$, then*

$$\int_\gamma P\, dx + Q\, dy = 0$$

*for all closed 1-forms $P\, dx + Q\, dy$.*

**Remark 1.1.** We don't necessarily need $\gamma$ to be $C^1$. It can, for example, be polygonal.

**Corollary 1.1.** *Let $\Omega$ be a domain. The following are equivalent:*

1. *$\Omega$ is simply connected.*
2. *If $f \in H(\Omega)$ satisfies $f(z) \neq 0$ for all $z \in \Omega$, then there exists $g \in H(\Omega)$ such that $f = e^g$.*

*Proof.* ($\implies$): Note that

$$\int_\gamma \frac{f'}{f}\, dz = 0.$$

So we can set

$$g(z) = \int_{z_0}^{z} \frac{f'(w)}{f(w)}\, dw.$$

($\impliedby$) If $|Omeag$ is not simply connected, let $f = z - a$ with $a \notin \Omega$. THen

$$\int_\gamma \frac{1}{z-a}\, dz \neq 0$$

for some $\gamma$. So there is no such $g$. □

**Corollary 1.2.** *Let $\Omega$ be a domain. The following are equivalent:*

1. *$\Omega$ is simply connected*
2. *For all harmonic $u : \Omega \to \mathbb{R}$ there exists a harmonic $v$ such that $u + iv \in H(\Omega)$.*

*Proof.* Assume $\Omega$ is imply connected. Then let $du = u_x\, dx + u_y\, dy$ and $*du = -u_y\, dx + u_x\, dy$. Condition 2 is equivalent to the existence of a harmonic $v$ such that $u_x = v_y$ and $u_y = -v_x$. Observe that $u$ is harmonic iff $*du$ is closed. So

$$\int_\gamma -u_y\, dx + u_x\, dy = 0$$

for all closed $\gamma$. Then let

$$v(z) = \int_{z_0}^{z} -u_y\, dx + u_x\, dy$$

this is well defined, and makes $v$ harmonic. □

**Example 1.1.** Let $a \notin \Omega$. Then

$$\int_\gamma \frac{1}{z-a} \neq 0$$

for some $\gamma$. If we set $u = \log|z-a|$, then $*du = \frac{1}{z-a}\, dz$.

## 1.2 Proofs of general Cauchy's theorem

Let's prove Cauchy's theorem.

*Proof.* There exists $R > 0$ such that $\gamma \subseteq \Omega_R = \Omega \cap \{z : |x| < R, |y| < R\}$. Let $\delta \leq \text{dist}(\gamma, \partial\Omega_R)/\sqrt{2}$. In particular, we can take $\delta = R/n$ for some $n \in \mathbb{N}$. We can pave the square $\{z : x \leq R, y \leq R\}$ by squares $S_j$ of side length $\delta$ with sides parallel to the axes. Now let $\Omega_\delta = (\bigcup_{s_j \subseteq \Omega_R} S_j)^o$, and let $\Gamma_\delta = \sum_{S_j \subseteq \Omega_R} \partial S_j$, after cancelling opposing arcs.

If $\zeta \in \Gamma_\delta$, there exists some $a \notin \Gamma_R$ such that $[a, \zeta], \cap \Omega_\delta = \varnothing$. Also, $\gamma \subseteq \Omega_\delta$. So for $z \in \gamma$,

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(z)}{z - -\zeta} \, d\zeta$$

because we can cancel all the boundaries of the squares to get the integral over $\Gamma_\delta$. Then, using Fubini's theorem,

$$\begin{aligned} \int_\gamma f(z) \, dz &= \int_\gamma \frac{1}{2\pi i} \int_{\gamma_\delta} \frac{f}{\zeta} \zeta - z \, d\zeta \, dz \\ &= \int_{\Gamma_\delta} f(\zeta) \underbrace{\frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - z} \, dz}_{=0} \, d\zeta \end{aligned}$$

where this term equals zero because the winding number is zero. $\square$

**Theorem 1.3** (Runge)**.** *Let $K \subseteq \mathbb{C}$ be compact, and let $K \subseteq U$, where $U$ is open. Let $f \in H(U)$. Then there exists a sequence $(R_n(z))_{n \in \mathbb{N}}$ of rational functions with poles outside $U$ such that*

$$\sup_K |f(z) - R_n(z)| \xrightarrow{n \to \infty} 0.$$

Runge's theorem implies the Cauchy integral formula. Here is a proof.

*Proof.* By polynomial division, we can write

$$R_n = P_n(z) + \sum_{k=1}^{M} \frac{c_k}{(z - z_k)^{n_k}},$$

so since $z_k \notin U$, we get that

$$\int_\gamma R_n(z) \, dz = 0.$$

By uniform convergence,

$$\int_\gamma f(z) \, dz = \int_\gamma R_n(z) \, dz = 0.$$

$\square$

How do you prove Runge's theorem? Use the same square method we used for the proof of Cauchy's theorem.[1]

[1]There is also a really interesting proof of Runge's theorem in my Functional Analysis (Math 255A) lecture notes. Although it seems to rely on Cauchy's theorem.